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The total rank and toral rank conjecture.

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Conjectures.

① [Halperin]. If a d-dim torus T acts freely on a compact CW complex X, then

TORAL RANK CONJECTURE

$$\sum_i \dim_{\mathbb{Q}} H^i(X,\mathbb{Q}) \geq 2^d.$$

② [Carlsson]. If $E = \mathbb{Z}/p^{\times d}$ acts ..., then $\sum_{i \geq 0} \dim_{\mathbb{F}_p} H^i(X, \mathbb{F}_p) \geq 2^d$.

③ R a local ring w/residue field k of dim d or $R = k[t_1, \ldots, t_d]$, $|t_i| = 2$. Let $F.$ be a bounded cplx of free R-modules with non-zero finite length homology, R semi-free dg-R-module ...

TOTAL RANK CONJECTURE.

Then,

$$\sum_{i \geq 0} \dim_k H^i(F \otimes_R k) \geq 2^d.$$

④ $F.$ a bounded complex of free modules over $\dfrac{\mathbb{Z}/p[t_1, \ldots, t_d]}{(t_1^p, \ldots, t_d^p)}$, $|t_i| = 0$. $\quad \mathbb{F}_p[E] \cong$, $E = \mathbb{Z}/p^d$.

Then, $\sum \dim_{\mathbb{F}_p} H_i(F) \geq 2^d$.

Rem. Algebraic conjectures imply the top. ones.

The top. ones imply the alg. ones <u>when E is a dga</u>.

Thm (Walker). For most R s.t. char $k = R/\mathfrak{m} \neq 2$, conjecture ③ is true for resolutions of finite length finite p.d. R-modules.

Already new for reg. local rings.

In fact, for any $F_\cdot$, with finite length

$$\sum \dim_k H^i(F \otimes_R k) \geq 2^d \frac{|\chi(F)|}{\sum \text{length}_i\, H_i(F)} \qquad \chi(F) = \sum (-1)^i H_i(F).$$

"Most" includes
- $R = k[[t_1, \ldots, t_d]]$, $|t_i| \neq 2$ and char $k \neq 2$. [redacted]
- char $R = p > 2$.
- R is a "Roberts ring", e.g. complete intersections.

R is a Roberts ring.

proof.

$$F \otimes_R F \underset{\substack{\uparrow \\ \text{Uses that } 2 \text{ is a unit.}}}{\simeq} \mathrm{Sym}^2 F \oplus \Lambda^2 F.$$

Since R is a Robert's ring,

$$\chi(\psi^2(F)) = 2^d \chi(F).$$

$\psi^2(F) = S^2 - \Lambda^2$, so

$$\underbrace{\chi(S^2(F)) - \chi(\Lambda^2(F))}_{\text{ignore negative terms}} = 2^d \chi(F).$$

In $K_0^{\mathfrak{m}}(R) \xrightarrow{\chi} \mathbb{Z}$

K-theory w/ supports.

$$2^d \chi(F) \leq \sum_{i \text{ even}} \text{length}\, H_i(S^2 F) + \sum_{i \text{ odd}} \text{length}\, H_i(\Lambda^2 F)$$

$$\leq \sum_i \text{length}\, H_i(F \otimes F).$$

$$\leq \sum_{p,q} \dim_k H_p(F \otimes_R k) \cdot \text{length}\, H_q(F_\cdot).$$

$$= \Big(\sum_p \dim_k H_p(F \otimes_R k)\Big)\Big(\sum \text{length}\, H_q(F_\cdot)\Big).$$

Get $\dfrac{2^d \chi(F)}{\sum \text{lengths}} \leq \sum \dim_k H_p(F \otimes_R k)$. Do the same with $F[1]$ to get $|\chi(F)|$.

Cor. If a d-dim torus T acts freely on a compact X,

$$\sum_i \dim_{\mathbb{Q}} H^i(X,\mathbb{Q}) \geq 2^d \frac{|\chi(X/T)|}{\sum \dim_{\mathbb{Q}} H^i(X/T,\mathbb{Q})}.$$

Cor. R as in the theorem, M non-zero R-module of f. length and f.pd,

$$\sum \beta_i(M) \geq 2^d.$$

$$\beta_i(M) = \dim_k \operatorname{Tor}_i^R(M,k).$$

Note. Buchsbaum-Eisenbud-Horrocks's conjecture predicts $\beta_i(M) \geq \binom{d}{i}$.

TOPOLOGICAL ANALOGUE?

The rest is joint w/ Iyengar.

Thm (Iyengar-Walker). R = reg local ring of dim $2n$, $R = k[t_1,\dots,t_{2n}]$, $|t_i| = 2$, resp. If char $k = 0$ or char $k > \frac{n+1}{2}$, then there exists a bounded cplx F of free R-modules with f length homology so that

$$\sum \text{length } H_i(F \otimes_R k) = \binom{2(n+1)}{n+1} < 2^{2n} \frac{4}{\sqrt{\pi(n+1)}}.$$

Thus, conjecture 3 is false in general, e.g., if char $k \neq 2$ and $d \geq 8$.

$$\binom{10}{5} \overset{?}{<} 2^8$$

$$\frac{\cancel{10}\cdot 9\cdot \cancel{8}\cdot 7\cdot \cancel{6}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}$$

$9 \cdot 2 \cdot 7 \cdot 2 < 256$

[illegible]

$18 \cdot 14$ ✓ ???

Construction. R reg. local. $\mathrm{Ext}^i_R(k,k) \cong \Lambda^i_k(x_1,\dots,x_n,y_1,\dots,y_n)$.

$|x_i| = |y_i| = 1$.

$\omega = \sum x_i \wedge y_i \in \mathrm{Ext}^2$

If $\operatorname{char} k = 0$ or $\operatorname{char} k > \frac{n+1}{2}$. Then,

$$\Lambda^i \xrightarrow{\ \omega\wedge -\ } \Lambda^{i+2}$$

is injective if $i \le n-1$, onto if $i \ge n+1$.

Proof in char 0: use Lefschetz on the cohomology of a torus.

Interpret $\omega : E \to E[2]$, $E \xrightarrow{\simeq} k$ a resolution.
Take the cone F of ω. So, $k \to k[2] \to F$.

And, $\dim_k \ker \omega + \dim_k \operatorname{coker} \omega = \binom{2(n+1)}{n+1}$.

Thm (Iyengar–Walker). If $\operatorname{char} k = p > 2$, $E = (\mathbb{Z}/p)^d$.
Then, there exists a finite complex of free kE-modules F.
s.t. $\sum \dim_k H_i(F) < 2^d$.

So, conjecture 4 is false if $\operatorname{char} k \neq 2$.